Few-Body Systems Suppl. 10, 105–114 (2008)

Few-
Body
Systems

Lorentz covariance of three-dimensional equations

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Abstract. We show how the invariance under the charge conjugation and CPT symmetry, present in the Bethe-Salpeter equation, is lost in the reduction to certain relativistic three-dimensional equations. This in particular leads to the breakdown of the standard Lorentz structure and renormalization procedures for the resulting single-particle propagators. We formulate the equal-time approximation of the Bethe-Salpeter equation in the form which manifestly satisfies the above symmetries, and apply it to the description of the pion-nucleon interaction in a dynamical model based on hadron exchanges. We also consider the one-body limit of various three-dimensional equations for the case of t- and u-channel one-particle-exchange potential.

1 Introduction

Quantum field theory (QFT) provides us with a suitable framework unifying principles of relativistic covariance and quantum mechanics. Unfortunately, any systematic calculation beyond perturbation theory is extremely complicated within QFT. On the other hand, in the ordinary quantum mechanics the scattering and bound state problems are well understood in terms of the Schrödinger or Lippmann-Schwinger equation. Its relativistic generalizations, referred to as *quasipotential* (QP) approximations to field theory, can therefore be very useful for practical studies of relativistic effects in the strongly interacting systems.

The QP equations can conveniently be obtained from the manifestly covariant four-dimensional Bethe-Salpeter (BS) equation by approximating the kernel in some way. This approximation involves an assumption about the singularities of the BS kernel, after which the integration over the 0-th component (time or energy) can easily be done explicitly, leading to the three-dimensional (3-D) equation. One of the first such reductions of the BS equation was studied by Salpeter [1], using the *instantaneous* approximation.

Since the covariant reductions can be done in infinitely many different ways it is desirable to establish certain criteria which would constrain the choice. For

instance, an important property one would like to have for a relativistic two-body equation is the *correct one-body limit*, which means that in the limit when one of the particles becomes infinitely heavy the two-body equation must reduce to the corresponding equation of motion of the light particle (e.g., the Klein-Gordon equation) in an external potential. The fact that the ladder BS equation does not have the correct one-body limit, created further motivation for the QP approach since the one-body limit can relatively simply be incorporated into a 3-D equation for the one-particle-exchange (OPE) potential. Some of the first equations of this type were suggested by Gross [2] and by Todorov [3]. Since then the one-body limit is regarded as an important criterion, even though not a very restrictive one, as many of the equations can be adjusted to satisfy it. In particular, Mandelzweig and Wallace [4] incorporated the one-body limit in Salpeter's instantaneous equation.

In this contribution we would like to demonstrate the importance of the constraint put by *charge conjugation* symmetry. Also, the one-body limit constraint will be examined at the one loop level for various OPE potentials. As an application, results of a quasipotential modelling of the pion-nucleon system will be presented as well.

2 The role of charge conjugation symmetry

To begin with, let us recall several basic definitions concerning the Lorentz group. A general Lorentz transformation L of a four-momentum is given by real (pseudo-)orthogonal tensor $\Lambda_{\mu\nu}$, and may belong to one of the following four domains:

$$\begin{aligned}
L_+^\uparrow &: \quad \det\Lambda = +1, \;\; \Lambda_{00} \geq +1, \\
L_-^\uparrow &: \quad \det\Lambda = -1, \;\; \Lambda_{00} \geq +1, \\
L_+^\downarrow &: \quad \det\Lambda = +1, \;\; \Lambda_{00} \leq -1, \\
L_-^\downarrow &: \quad \det\Lambda = -1, \;\; \Lambda_{00} \leq -1.
\end{aligned} \tag{1}$$

From these, only transformations $L_+^\uparrow$ form a group by themselves, called the proper orthochronous Lorentz group. The other domains do not contain the unity element, however their multiplication with $L_+^\uparrow$ may form a group. Fields and corresponding Green functions are transformed according to unitary representations of the Lorentz group. Let us remark, that the full Lorentz group transformations [which include the proper continuous transformations, and the (anti-)unitary transformations of parity and time reversal] do not connect Green functions defined for positive energy (upper light-cone) with those for negative energy (lower light-cone). In other words, applying such transformations to the momentum-space Green functions can induce only $L_+^\uparrow$ and $L_-^\uparrow$ transformations of the relevant four-momenta. To be able to induce $L_+^\downarrow$ and $L_-^\downarrow$ transformations of the four-momenta, one needs to include charge conjugation in addition to the above mentioned Lorentz transformations.

In considering some existing relativistic 3-D equations, we find that they in general do not yield the correct Lorentz structure. For example, the calculated self-energy of a spin-1/2 particle *does not* have the following form,

$$\Sigma(\not{P}) = \not{P}A(P^2) + B(P^2), \tag{2}$$

where A and B are scalar functions of the invariant P^2 only. At first this is surprising, naively we would expect form (2) to come out in any covariant formalism. However, Eq. (2) holds only if there is a symmetry under all Lorentz transformations of the four-momenta. Therefore, in order to obtain the self-energy consistent with Eq. (2), the relativistic equation in question should be covariant under the Lorentz group and charge conjugation.

The four-dimensional BS equation, of course, preserves the standard structure, such as Eq. (2). The symmetry can obviously be lost in doing the QP reduction. To illustrate this, consider the example of a scalar self-energy, given by

$$\Sigma(P^2) = i\int \frac{\mathrm{d}^4q}{(2\pi)^4}\,\frac{\Phi(q^2,P^2,P\cdot q)}{[(\frac{1}{2}P-q)^2-m^2+i\varepsilon]\,[(\frac{1}{2}P+q)^2-m^2+i\varepsilon]}, \tag{3}$$

where P is the relevant four-vector, Φ is an "interaction function" which corresponds to the product of the two vertex functions, and which may also have some particle propagation poles.

We can immediately see that Eq. (3) is a function of P^2 only: a sign change of P can be absorbed by a change of the loop variable q to $-q$. In a QP description this substitution in general cannot be applied in view of the constraint in q_0. To see what happens then, consider the poles of the integrand of Eq. (3) in the complex q_0 plane.

There are four poles (two in the upper and two in the lower half-plane) coming from the propagators in the two-particle Green function:

$$q_0 = \pm\tfrac{1}{2}P_0 - \sqrt{m^2+(\tfrac{1}{2}\boldsymbol{P}\mp\boldsymbol{q})^2} + i\varepsilon, \text{ and } q_0 = \pm\tfrac{1}{2}P_0 + \sqrt{m^2+(\tfrac{1}{2}\boldsymbol{P}\mp\boldsymbol{q})^2} - i\varepsilon.$$

We can see that a simultaneous sign reflection of P_0 and q_0 interchanges the poles of the upper half-plane with the poles of the lower half-plane. The same symmetry exists for the singularities of Φ. Therefore, in order for Σ to be even in P_0 the integration over q_0 must be independent of the choice of the half-plane where we close the contour. In performing a 3-D reduction, however, one usually neglects the contribution of certain poles, hence the result becomes dependent on the contour. In that case Σ is not anymore an even function of P_0, consequently it cannot be a function of P^2 only. In this sense the standard Lorentz structure of the self-energy is violated. (An example of the QP prescription, which is covariant under the Lorentz group but violates the charge conjugation symmetry, is the spectator approximation of Gross [2]. In this approximation, one of the particles inside the loop is restricted to its mass shell, therefore only a single pole is taken in calculating the q_0 integral.)

Similar arguments apply for the spin-1/2 particle self-energies. Consider the dressed fermion propagator given by

$$S(\not{P}) = [\not{P} - m - \Sigma(\not{P}) + i\epsilon]^{-1}\,, \tag{4}$$

where $\Sigma(\not{P})$ is the self-energy. For simplicity we work in the c.m. frame, where $P = (P_0, \mathbf{0})$. In this frame the Dirac structure of the self-energy can be represented as

$$\Sigma(P_0) = \Sigma_+(P_0)\gamma_+ + \Sigma_-(P_0)\gamma_-, \tag{5}$$

where $\gamma_\pm = \frac{1}{2}(I \pm \gamma_0)$. A similar decomposition holds for the propagator:

$$S(P_0) = S^{(+)}(P_0)\gamma_+ + S^{(-)}(P_0)\gamma_-, \tag{6}$$

with $S^{(\pm)}(P_0) = \pm[P_0 \pm (-m - \Sigma_\pm(P_0) + i\epsilon)]^{-1}$. Obviously, $S^{(+)}$ corresponds to the positive and $S^{(-)}$ to the negative energy-state propagation.

It is easy to see that, if the self-energy can be written in the general covariant form (2), then the following identity holds,

$$\Sigma_r(P_0) = \Sigma_{-r}(-P_0), \quad r = \pm 1, \tag{7}$$

and vise versa (in the c.m. frame). This identity is particularly useful to test numerically Eq. (2) in models based on QP equations which are usually solved for partial waves in the c.m. system.

Performing the standard renormalization procedure by subtracting the counter-term: $Z_2(m_0 - m) + (1 - Z_2)(\not{P} - m)$, where m_0 is the bare mass, and Z_2 is the field renormalization constant, we find that the on-shell renormalization scheme requires

$$\begin{aligned} Z_2(m_0 - m) &= \Sigma_+(m) = \Sigma_-(-m), \\ 1 - Z_2 &= \left.\frac{\partial \Sigma_+(P_0)}{\partial P_0}\right|_{P_0=m} = -\left.\frac{\partial \Sigma_-(P_0)}{\partial P_0}\right|_{P_0=-m}. \end{aligned} \tag{8}$$

Obviously, it is not possible to satisfy these relations if Eq. (7) is violated. In other words, the violation of the extended Lorentz symmetry leads to the different renormalization of the positive and negative energy states. This can be understood, as the violation of the charge conjugation symmetry in a Lorentz-covariant framework implies violation of CPT symmetry.

To recapitulate, relativistic equations obtained from the BS equation via the 3-D reduction which discriminates between the positive and negative energy poles (e.g. by putting particles on-shell, or using the positive energy projection operators) lead to results which do not have the standard Lorentz structure, even if the symmetry under the full Lorentz group remains intact. Such equations necessarily violate the charge conjugation and CPT symmetries, and thus lead to the breakdown of the usual renormalization procedures which rely on constructing the counter-terms from a CPT invariant Lagrangian. Also, one then cannot use the standard covariant arguments to construct the transformation properties of the calculated amplitudes (as well as any other functions involving loop corrections), which for instance is needed to incorporate the basic interaction in more particle systems.

3 Manifestly covariant three-dimensional equation

One of the ways to perform a 3-D reduction consistent with charge conjugation and unitarity is by removing the poles of the interaction in q_0 complex plane, while treating exactly the poles of the two-particle propagator. This procedure is realized in the equal-time (ET) approximation [4, 5]. In this approximation the poles are removed from the interaction piece by fixing the relative-energy variable q_0 in some way. Most frequently the constraint $q_0 = 0$, or its Lorentz-invariant generalization, $P \cdot q = 0$, is used. Moreover, the two-particle propagator is sometimes modified to include approximately the crossed graphs [4, 6].

On the other hand, it is well known that the $P \cdot q = 0$ constraint is troublesome in the inelastic or more-particle problems, see, e.g., the introductory remarks in Refs. [7, 8]. The weak point resides in the fact that the constraint is embedded through a δ-function. In the following we formulate a 3-D formulation which exhibits manifest Lorentz covariance, does not make use of δ-functions, and for the elastic two-body problem is equivalent to the ET approximation.

Recall the two-particle Bethe-Salpeter equation in the momentum-space:

$$T(p', p) = V(p', p) + i \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \, V(p', q) \, G(q) \, T(q, p), \tag{9}$$

where we assume p', p and q are the relative four-momenta of the final, initial and intermediate state, respectively. To transit to the 3-D formulation we impose the condition that the interaction is insensitive to the off-shellness along the direction defined by unit four-vector n_μ. For the two-body case this means that V and T entering the scattering equation depend on the projections of the relative four-vectors onto a 3-D hyperplane orthogonal to n_μ. Defining the projection operator: $O_{\mu\nu} = \mathrm{g}_{\mu\nu} - n_\mu n_\nu$, we write the corresponding equation as follows:

$$T(\tilde{p}', \tilde{p}) = V(\tilde{p}', \tilde{p}) + i \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \, V(\tilde{p}', \tilde{q}) \, G(q) \, T(\tilde{q}, \tilde{p}), \tag{10}$$

where $\tilde{p}_\mu = O_{\mu\nu} p^\nu$, and similarly for $\tilde{p}'$, $\tilde{q}$. Equation (10) is manifestly covariant, and on the other hand it can easily be reduced to the 3-D form. For example, let us choose the frame where $n = (1, 0, 0, 0)$, and therefore V and T are independent of the 0-th component of relative momenta (since any scalar product will depend only on the spatial components, e.g., $\tilde{q} \cdot \gamma = -\boldsymbol{q} \cdot \boldsymbol{\gamma}$). The integration over q_0 in Eq. (10) can now be readily done leading to the 3-D equation.

Obviously, the newly introduced four-vector n will enter the final covariant forms. To prevent this dependence one may choose it along some physical four-momentum, for instance the total momentum of the system, i.e.,

$$n_\mu = P_\mu / \sqrt{P^2}. \tag{11}$$

It is then easy to see that for the two-body elastic scattering Eq. (10) in the c.m. system becomes equivalent to the usual ET approximation.

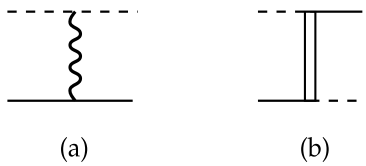

Figure 1. The t-channel (a) and u-channel (b) exchange potentials.

4 Box graphs and the one-body limit

Analyzing the box and the crossed-box graphs in QFT, for a neutral particle exchange, Gross revealed a cancellation among various pole contributions, and proved that the only pole which survives in the limit is that of the heavy particle in the intermediate state of the box graph [2, 9]. This led him to formulate the spectator equation where the heavy particle is on the mass-shell. Obviously, the heavier is the spectator, the closer should the Gross (spectator) equation be to the QFT result. Therefore, for instance the πN system would seem to be a particularly good application for this equation, since the nucleon is much heavier than the pion. Recently, however, Gross and Surya, applying the spectator equation to πN system, have argued that the light particle (the pion) must be taken as the spectator [10, Sec. II.A]. Studying the box and the crossed-box graphs at threshold, they have conjectured that "the essential difference is the mass of the exchanged particle". We have examined their conjecture, studying the graphs for more general situations, and find that the argument should be related more to the type of the OPE potential, rather than to the mass of the exchanged particle.

Namely, we consider two types of the potentials, see Fig. 1: (a) t-exchange potential, and (b) u-exchange potential. (We shall refer to the dashed line particle as to pion and the solid line as to nucleon with corresponding masses m_π and m_N, the exchange particle mass is denoted as μ.) Substituting these

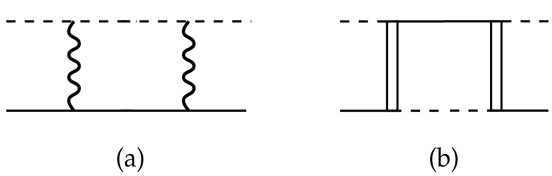

Figure 2. The box graphs obtained by iterating once the potentials of Fig. 1.

potentials into the scattering equation, Fig. 4, and iterating once, we obtain the box graphs depicted in Fig. 2 (a) and (b), respectively. Note that in QFT, due to the crossing symmetry, one in addition has the corresponding crossed-box graphs.

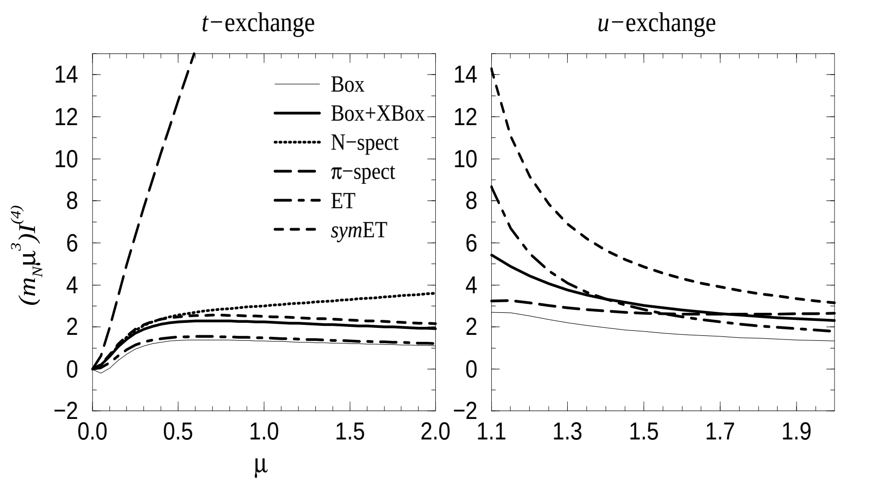

Figure 3. Results for $m_N = 1$, $m_\pi = 0.01$, $\sqrt{s} = 1.1$ as the function of the exchange particle mass.

We have calculated such box and crossed-box graphs in 4-D field theory numerically and compared with the box graph calculation within various quasipotential formulations. Namely, the nucleon and the pion *spectator* [2, 9, 10], the *equal-time* [5], and the *symmetrized equal-time* [4, 6].

The results of these calculations for the case close to the one body limit (the nucleon is much heavier that the pion) is plotted in Fig. 3, as a function of the exchange particle mass.[1] The energy is fixed slightly above the threshold, $\sqrt{s} = 1.1 m_N$, and $t = 0$. One can see that for the t-exchange potential the one-body limit is achieved in the symmetrized ET formulation independently of the mass of the exchanged particle. The nucleon spectator indeed deviates from the limit for large μ, however the pion spectator does not produce a better result in this situation.

On the other hand, in the u-exchange case, both the nucleon spectator and symmetrized ET disagree substantially with the QFT result (the spectator calculation is an order of magnitude larger and hence beyond the scale of the figure). The pion spectator is in a much better agreement. Thus, we conclude that the difference between the NN and πN situation encountered by Gross and Surya [10] appears due to the different *type of the potential*.

It would be interesting to see if there is a possibility to develop a prescription which would give the proper limit in both the t- and u-exchange cases. It should be emphasized though, that the one-body limit situation is physically very different for the two cases: for the t-exchange potential it corresponds to the light particle moving in an external potential of the heavy particle, while in the

[1] Note that we multiply the results by $\mu^3 m_N$ in order to obtain reasonable values for various limiting cases.

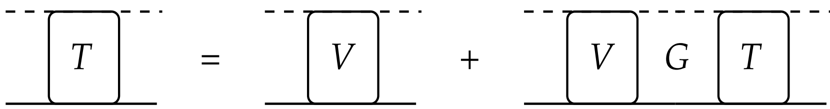

Figure 4. Diagrammatic form of a relativistic two-body scattering equation.

u-exchange case the heavy particle obviously does not act as a static external source, and therefore there seems to be no correspondence to any one-body situation.

5 πN scattering

We have studied the ET approximation of the BS equation in a dynamical model for πN scattering. The corresponding equation, Fig. 4, is solved for the πN partial-wave amplitudes with the OPE potential represented by $N(938)$, $N^*(1450)$, $\Delta(1232)$, $D_{13}(1525)$, $S_{11}(1555)$, $\rho(770)$ and $\sigma(550)$ exchanges, see Fig. 5. The model is very close to the one presented earlier [11], even though presently we have used a different form of the $\pi N \Delta$ coupling [12], and D_{13} and S_{11} exchanges are included in addition. The latter has a considerable effect only in the S_{11} partial wave.

The model parameters (coupling constants, resonance and cutoff masses) were adjusted to reproduce the low-energy quantities, such as scattering lengths, volumes and ranges, and the energy behavior of the phase-shifts. The resulting description of the phase-shifts up to 600 MeV pion kinetic lab energy is depicted in Fig. 6.

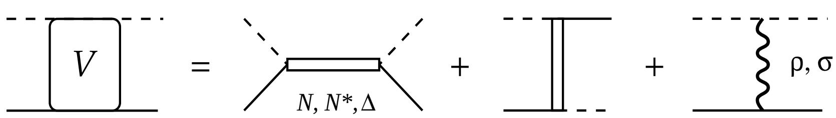

Figure 5. The tree-level πN potential, the driving force of the scattering equation.

6 Conclusions

The relativistic scattering and bound state problems are often formulated in terms of a 3-D (or quasipotential) relativistic equation of the Lippmann-Schwinger type. Such equations can be obtained from the manifestly covariant (3+1)-dimensional Bethe-Salpeter equation by integrating out the time variable in some approximate way, thus performing the so called 3-D reduction. Adopting the 3-D formulation in favor of the 4-D one leads to major technical simplifications, since the field-theoretical BS kernel may, in principle, contain many singularities in the time variable. However, the charge conjugation symmetry can easily be violated in performing such a reduction. On the other hand, it plays an important role in obtaining the standard Lorentz structure

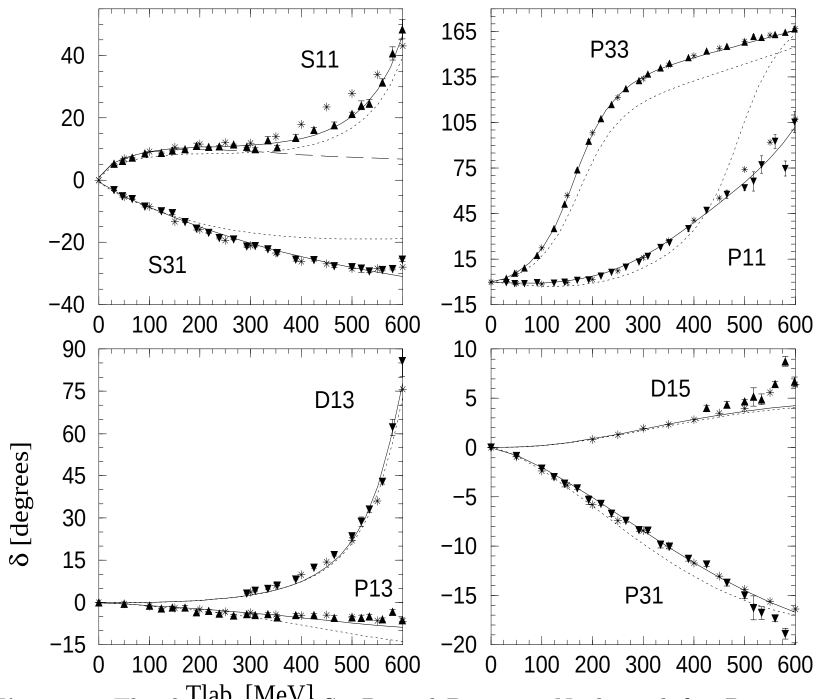

Figure 6. The description of S-, P- and D-wave πN phase-shifts. Data points are from the SM95 [13] (triangles) and KH80 [14] (stars) partial-wave analyses. Solid lines represent the model solution. Dotted lines represent the calculation where the principal value part of the rescattering integrals is switched off (i.e.,the K-matrix approximation with the same set of parameters). Dashed line for the S_{11} shows the calculation when the pole contribution of the S_{11} resonance is switched off.

of the loop corrections. Therefore, the equations which respect charge conjugation symmetry are preferable, and thus the choice among the infinite number of possible relativistic 3-D equations is somewhat restricted in this way.

We have presented a 3-D reduction which is manifestly covariant under the complete set of Lorentz transformations as well as charge conjugation. The two-body equation, obtained by using this reduction, in the c.m. system is equivalent to the Salpeter equation.

We have studied the one-body limit for several 3-D equations with the OPE potential. In the limit a large qualitative difference is observed between the situation when the potential in question has the form of t- or u-channel exchange. The 3-D equations, such as the nucleon spectator [2, 9] and the symmetrized ET [4, 6], developed to satisfy the one-body limit for the t-type exchange potential, have a poor agreement with the exact calculation if the u-type exchange potential is used.

The pion spectator approximation describes the u-exchange case better, but fails in the other case. Therefore, in the situation where both types of

the potential are present, either of the spectator equations cannot be justified. Analyzing the πN situation with realistic parameters we find that the ET type of prescriptions can be fairly close to the QFT answer for both types of the potential, and, hopefully, is a reasonable dynamical framework in this case. We have therefore applied the ET approximation of the BS equation to the description of the πN scattering.

Acknowledgments

We thank Dr. F. Coester for illuminating discussions on Lorentz covariance. One of us (V.P.) had the pleasure to discuss with Professor V. Mandelzweig during the conference.

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